Analytic continuation and fixed points of the Poincaré mapping for a polynomial Abel equation

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Abstract

We consider an Abel differential equation $y' = p(x)y^2 + q(x)y^3$ with $p(x)$, $q(x)$ – polynomials in x. For two given points a and b in $\mathbb{C}$, the "Poincaré mapping" of the above equation transforms the values of its solutions at a into their values at b. In this paper we study global analytic properties of the Poincaré mapping, in particular, its analytic continuation, its singularities and its fixed points (which correspond to the "periodic solutions" such that $y(a) = y(b)$). On one side, we give a general description of singularities of the Poincaré mapping, and of its analytic continuation. On the other side, we study in detail the structure of the Poincaré mapping for a local model near a simple fixed singularity, where an explicit solution can be written. Yet, the global analytic structure (in particular, the ramification) of the solutions and of the Poincaré mapping in this case is fairly complicated, and, in our view, highly instructive. For a given degree of the coefficients we produce examples with an infinite number of *complex* "periodic solutions" and analyze their mutual position and branching. Let us remind that Pugh's problem, which is closely related to the classical Hilbert's 16th problem, asks for the existence of a bound to the number of *real* isolated "periodic solutions".

1 Introduction.

In this paper we start an investigation of the global analytic properties of the "Poincaré mapping" ϕ for an Abel differential equation of the form

$$y' = p(x)y^2 + q(x)y^3.$$

For two given points a and b, ϕ transforms the values $y(a)$ of the solutions $y(x)$ of this equation at a into their values $y(b)$ at b. A more accurate definition is given in Section 3 below.

We follow here the approach of the classical Analytic Theory of Differential Equations (see [23, 13, 14, 16]) and consider the Abel equation, its solutions and its Poincaré mapping in the *complex domain*. Consequently, the main questions under investigation are fixed and movable singularities, analytic continuation and ramification of the functions involved.

Below we always restrict ourselves to the case of a *polynomial* Abel equation

$$y' = p(x)y^2 + q(x)y^3, \quad y(0) = y_0 \tag{1.1}$$

with $p(x), q(x)$ – polynomials in the complex variable x of the degrees d_1, d_2 respectively.

The following two problems for Equation (1.1) have been intensively studied (see [4]-[8]):

a. For given a, b is it possible to bound the number of *real* solutions $y(x)$ of (1.1), satisfying $y(a) = y(b)$, in terms of the degrees d_1, d_2 only?

b. Is it possible to give explicit conditions on p and q for $y(a) \equiv y(b)$ for all the solutions of (1.1)?

These two problems are well known to be closely related to the classical Hilbert's 16th problem (second part) and Poincaré's Center-Focus problem for polynomial vector fields on the plane.

Adopting the standard terminology in these problems, we shall call solutions of (1.1), satisfying $y(a) = y(b)$, the "closed" or the "periodic" ones, and the conditions on $p, q,\ a, b$ for $y(a) \equiv y(b)$ the center conditions.

Abel equations were first investigated and studied by Abel himself as natural extensions of Ricatti equations. Abel found several examples which are integrable ([1]). Then this list was enriched by Liouville. Classical references are [1, 19, 13, 14], and the modern references [16, 20, 21] have been instrumental for us.

The main motivation for our study comes from these classical examples of the polynomial Abel equation which can be solved explicitly. Moreover, we mostly (but not always) restrict ourselves to one case where the first integral is rational. However, the global analytic structure (in particular, the ramification) of the solutions and of the Poincaré mapping in this example turns out to be fairly complicated, and, in our view, highly instructive. We study the singularities and the branching of the solutions and of the Poincaré mapping. In particular, for a fixed degree of the coefficients we produce, varying the parameters, examples with arbitrarily many and with an infinite number of "periodic solutions". We analyze the mutual position and branching of these periodic solutions.

Our attempt to better understand the global structure of the Poincaré mapping for the polynomial Abel equations was motivated also by the recent progress in the investigation in [4]-[8], [3, 9, 24] of the Center-Focus problem for (1.1). As it was mentioned above, this problem is to give explicit conditions on p and q for the Poincaré mapping on a, b to be identical, i.e. for $y(a) \equiv y(b)$. In particular, in [4]-[8] the Moment and the Composition conditions, providing a close approximation of the center conditions, have been introduced. On this base in [3] the complete "local center conditions" have been obtained, and the "local Bautin ideal" has been computed for the Poincaré mapping ϕ, while in [9] similar conditions "at infinity" have been

found.

Via Bautin's approach [2], further developed in [11, 12, 27] the knowledge of the Bautin ideal of the Poincaré mapping ϕ allows one to produce "semi-local" bounds on the fixed points of ϕ. In other words, we get a fairly accurate control of the fixed points inside the disk of convergence of the Taylor series of ϕ at the origin. Let us remind that the problem of the *global control* of real fixed points of ϕ is very closely related to the Hilbert 16th problem of counting limit cycles of the plane vector-fields.

However, the methods of [11, 12, 27] are at present absolutely limited to the disk of convergence of ϕ. Any attempt to "globalize" the information produced by these methods will require a much better understanding of the global analytic nature of ϕ, in particular, of its analytic continuation, its singularities and its ramification structure. In this paper we start an investigation in this direction.

The paper is organized as follows:

In Section 2 we reprove the classical results of [23] which provide the description of singularities of the solutions of (1.1). Our proof is somewhat more "quantitative" than the classical one, providing an accurate estimates of the domains and the parameters involved. We also prove some lemmas relating the position of the singularities of the solutions of (1.1) with the initial values of these solutions.

On this base, in Section 3, we give an accurate definition of the Poincaré mapping ϕ, discuss the problem of the analytic continuation of ϕ, and give a constructive procedure of this analytic continuation, based on the path deformation following the moving singularities of the solutions. In Section 4 we describe typical singularities of the Poincaré mapping ϕ. This completes our general description of the Poincaré mapping for Abel equation.

In Section 5 we discuss a local model of Abel equation near a simple fixed singularity, and produce its explicit solutions.

In Section 6 we analyze the singularities and the ramification of the solutions and of the Poincaré mapping and we combine the results to analyze the geometry of the periodic solutions.

2 The Abel equation

Below we shall always assume that the functions $p(x)$ and $q(x)$ in the Abel equation (1.1) are polynomials in x with complex coefficients. Most of the results below remain valid for $p(x)$ and $q(x)$ - much more general analytic functions, but our assumption simplifies a presentation.

Let $a \in \mathbb{C}$. Denote by $y(y_a, x)$ the solution of the equation (1.1), satisfying $y(y_a, a) = y_a$. By the uniqueness and existence results for ordinary differential equations, the solution $y(y_a, x)$ exists in a certain neighborhood of a and is there a regular complex analytic function of the complex argument x. However, an analytic continuation of $y(y_a, x)$ may lead to singularities.

The classical result of Painlevé [23] shows that the "movable" singularities of the solutions $y(y_a, x)$ must be "algebroid". Moreover, following the proof of Painlevé (see, for example [23, 13, 16]), one can easily show that at each movable singular point x_0, $y(y_a, x)$ behaves as $\frac{1}{\sqrt{x-x_0}}$. In order to relate singularities of y with those of the Poincaré mapping ϕ we need more detailed information on the position of singularities, on their dependence on the initial values, etc., than is usually given. So we reprove in the special case of the equation (1.1) the classical results, providing all the required estimates.

Notice that $y \equiv 0$ is a solution of (1.1). It follows, in particular, that as $y_a \to 0$ all the singularities of $y(y_a, x)$ tend to infinity. Below we make this remark more precise.

Another remark is that, as we shall see below, the problematic points of the equation (1.1) are zeroes of $q(x)$. We denote these zeroes $x_1, \cdots, x_m$ and always distinguish between the "fixed" singularities of y at $x_1, \cdots, x_m$

and the "movable" singularities of y, which occur at points different from $x_1, \cdots, x_m$.

2.1 Domain of regularity of the solutions

The following assumptions will be preserved along the rest of this section: $p(x)$ and $q(x)$ are polynomials of degree m in x, with $\|p\|,\ \|q\| \leq K$. The norm of a polynomial is defined here as the sum of the absolute values of its coefficients. Let $a \in \mathbb{C}$. Denote, as above, by $y(y_a, x)$ the solution of the equation (1.1), satisfying $y(y_a, a) = y_a$.

Lemma 2.1 *Let $a \in \mathbb{C},\ y_a \in \mathbb{C}$ be given. Then the solution $y(y_a, x)$ exists in a disk $D_\rho(a)$ centered at a. Here $\rho = \rho(|a|, |y_a|)$ is a positive explicitly given function of its arguments, which for $|y_a|$ big satisfies*

$$\rho(|a|, |y_a|) \geq C_1 (4K|a|^m |y_a|^2)^{-1}.$$

For $|y_a|$ small ρ satisfies

$$\rho(|a|, |y_a|) \geq C_2 \left(\frac{1}{2K|y_a|} \right)^{\frac{1}{m+1}}.$$

In particular, ρ tends to infinity as $|y_a|$ tends to zero.

The solution $y(y_a, x)$ is bounded in the disk $D_\rho(a)$ by $\hat{y}(|y_a|, |a|, |x - a|)$, with $\hat{y}$ an explicitly given function of its arguments, satisfying

$$\hat{y}(|y_a|, |a|, t) \leq C_3(|y_a|, |a|)(\rho - t)^{-\frac{1}{2}}.$$

Proof: For each $x \in \mathbb{C}, |p(x)| \leq K|x|^m, |q(x)| \leq K|x|^m$. Hence, the right hand side of (1.1) is bounded in absolute value by $K|x|^m(|y|^2 + |y|^3)$. Therefore we get the following differential inequality:

$$\frac{d|y|}{dv} \leq K|x|^m(|y|^2 + |y|^3). \tag{2.1}$$

Here $\frac{d}{dv}$ denotes a directional derivative at x in any (normalized) direction v in the complex plane.

Now consider a straight line ℓ in $\mathbb{C}$, passing through a and let t be a normalized parameter along ℓ, with $t = 0$ at a. Since for a running point $x(t) \in \ell,\ |x(t)| \leq |a| + t$ we get by (2.1) the following differential inequality with respect to t:

$$\frac{d|y|}{dt} \leq K(|a| + t)^m(|y|^2 + |y|^3). \tag{2.2}$$

Denote by $\tilde{y}(t) = \tilde{y}(|a|, |y_a|, t)$ the solution of the differential equation

$$\frac{dy}{dt} = K(|a| + t)^m(y^2 + y^3), \tag{2.3}$$

satisfying for $t = 0$ the initial condition $\tilde{y}(0) = |y_a|$. Then by (2.2) for each $t \geq 0$ we have $|y(y_a, x(t))| \leq \tilde{y}(t)$.

It remains to compute $\tilde{y}(t)$. Separating variables we obtain

$$\frac{dy}{y^2(y+1)} = dy(\frac{-1}{y} + \frac{1}{y^2} + \frac{1}{y+1}) = K(|a| + t)^m,$$

which gives after integration the following implicit equation for the solution $\tilde{y}(t)$, where we denote the function $ln(1 + \frac{1}{y}) - \frac{1}{y}$ by F(y):

$$F(y) = F(|y_a|) + \frac{K}{m+1}((|a| + t)^{m+1} - |a|^{m+1}). \tag{2.4}$$

The function $F(y)$ for y positive is a negative strictly increasing function, tending to $-\infty$ as y tends to zero, and approaching zero from below as y tends to ∞. The inequality $\frac{1}{y^2(y+1)} < \frac{1}{y^3}$ shows that $F(y) > -\frac{1}{2y^2}$. To bound $F(x)$ from above, define $h(y) = \frac{1}{2y^3}$ for $y \geq 1$, and $h(y) = \frac{1}{2y^2}$ for $y \leq 1$. We have $h(y) < \frac{1}{y^2(y+1)}$, and therefore $F(x) < H(y)$, where $H(y) = \int h(y)dy = -\frac{1}{4y^2}$ for $y \geq 1$ and $H(y) = -\frac{1}{2y} + \frac{1}{4}$ for $y \leq 1$. Finally we get the following bounds from two sides for $F(y)$:

$$-\frac{1}{2y^2} < F(y) < H(y). \tag{2.5}$$

Solving the equation (2.4) and taking into account the lower bound in (2.5) we get

$$\tilde{y}(t) < [-2(F(|y_a|) - \frac{K}{m+1}((|a|+t)^{m+1} - |a|^{m+1}))]^{-\frac{1}{2}}.$$

Now applying the upper bound in (2.5) we finally obtain the following inequality:

$$\tilde{y}(t) < \hat{y}(t) = [-2(H(|y_a|) - \frac{K}{m+1}((|a|+t)^{m+1} - |a|^{m+1}))]^{-\frac{1}{2}}. \tag{2.6}$$

Taking into account an explicit definition of the function H given above, we obtain

$$\hat{y}(t) = [\frac{1}{4|y_a|^2} - \frac{K}{m+1}((|a|+t)^{m+1} - |a|^{m+1}))]^{-\frac{1}{2}},\ |y_a| \geq 1,$$

and

$$\hat{y}(t) = [\frac{1}{2|y_a|} - \frac{1}{4} - \frac{K}{m+1}((|a|+t)^{m+1} - |a|^{m+1}))]^{-\frac{1}{2}},\ |y_a| \leq 1. \tag{2.7}$$

The function $\hat{y}(t) = \hat{y}(|a|, |y_a|, t)$ grows with t. For $t = 0$ it takes value $2|y_a|$, if $|y_a| \geq 1$, and it takes value $(\frac{1}{2|y_a|} - \frac{1}{4})^{-\frac{1}{2}}$, if $|y_a| \leq 1$ (which is of order $\sqrt{2|y_a|}$ for small $|y_a|$). This function is finite until the expression in the parentheses remains positive. This gives the following expression for the radius of the disk of existence of the solution:

$$\rho(|a|, |y_a|) = |a|([1 - \frac{m+1}{K|a|^{m+1}} H(|y_a|)]^{\frac{1}{m+1}} - 1). \tag{2.8}$$

For $|y_a|$ big this gives us the following expression:

$$\rho(|a|, |y_a|) \simeq (4K|a|^m |y_a|^2)^{-1}. \tag{2.9}$$

For $|y_a|$ small we get

$$\rho(|a|, |y_a|) \simeq \left(\frac{1}{2K|y_a|}\right)^{\frac{1}{m+1}}. \tag{2.10}$$

This completes the proof of Lemma 2.1.

Corollary 2.1 *$y(y_a, x)$ is regular in the disk D_R of radius R centered at the origin, with R growing as $\left(\frac{1}{2K|y_a|}\right)^{\frac{1}{m+1}}$, as $|y_a|$ tends to zero.*

2.2 Singularities of $y(y_a, x)$

In this subsection we reprove the classical results on the structure of singularities of $y(y_a, x)$, stressing the explicit estimates of the size of the domains, where the results are valid. The assumptions on p and q and the notations remain the same as in subsection 2.1.

Lemma 2.2 *If $x_0 \in \mathbb{C}$ is a singular point of the solution $y(x)$ of (1.1), then y tends to infinity as x tends to x_0.*

Proof: If $\lim_{x \to x_0} y(x) \neq \infty$, then there is a constant $c > 0$ and a sequence x_i converging to x_0 such that $|y(x_i)| \leq c$ for each i. Applying Lemma 2.1 to one of the points x_i, taken as a, we obtain that $y(x)$ is regular inside a disk around x_i, of a certain radius $\rho > 0$ which does not depend on i. Taking x_i sufficiently close to x_0, we conclude that y is regular at and around x_0. This contradiction proves the lemma.

Now we give an analytic description of the movable singular points of the solutions of (1.1). As we have already mentioned in the Introduction, the classical result of Painlevé [23, 13, 16] shows that the "movable" singularities of the solutions $y(y_a, x)$ of the equation (1.1) must be "algebroid". Moreover, following the proof of Painlevé (see, for example [23, 13, 16]), one can easily show that at each movable singular point x_0, $y(y_a, x)$ behaves as $\frac{1}{\sqrt{x-x_0}}$. However, in order to relate singularities of y with those of the Poincaré mapping ϕ we need more accurate estimates, than are usually given, and in particular, we have to describe the behavior of the movable singular points of the solutions of (1.1) as the function of the initial value y_a. So we reprove below the classical result of Painlevé (in the special case of the equation (1.1)), providing the required estimates.

To simplify the statement of the results below, let us introduce some notations. Assume that $x_0 \in \mathbb{C}$ is given, x_0 different from the zeroes $x_1, \cdots, x_m$ of $q(x)$. Let $\eta(x_0) = |q(x_0)| > 0$. Put $R(x_0) = 2(|x_0| + 1)$ and define $r(x_0) = min\left(\frac{1}{4}R, \frac{\eta}{2m(K+1)R^{m-1}}\right)$, where K as above is the maximum of the norms $\|p\|, \|q\|$. Finally, let us define $M(x_0) = \frac{4m(K+1)R^m}{\eta}$ and put $\delta(x_0) = \frac{1}{M}$ and $c(x_0) = \frac{4}{M\eta} = \frac{1}{m(K+1)R^m}$.

Theorem 2.1 *For any $x_0 \in \mathbb{C}$, x_0 different from the zeroes $x_1, \cdots, x_m$ of $q(x)$, there is a unique solution $y(x) = y^{[x_0]}(x)$ of (1.1) with a singularity at x_0. This solution has an algebraic ramification of order 2 at x_0. In a neighborhood of its singular point x_0, the solution $y^{[x_0]}(x)$ of (1.1) is given by the Puiseux series*

$$y^{[x_0]}(x) = \frac{c(x_0)}{(x - x_0)^{1/2}} \cdot (1 + \sum_{k=1}^{\infty} c_k(x_0)(x - x_0)^{\frac{k}{2}}), \tag{2.11}$$

converging for $|x - x_0| \leq r(x_0)$, with the coefficients $c(x_0)$, $c_k(x_0)$ univalued analytic functions in $\mathbb{C} \setminus \{x_1, \cdots, x_m\}$, satisfying there $|c_k(x_0)| \leq \delta(x_0)(r(x_0))^{-k}$.

Proof: The proof of Theorem 2.1 takes the rest of the present subsection. In the course of the proof we stress some intermediate steps which will be used later as lemmas, propositions, etc.

After the change of the dependent variable $u = \frac{1}{y}$ the equation (1.1) takes the form

$$\frac{du}{dx} = -p(x) - q(x)\frac{1}{u} = -\frac{p(x)u + q(x)}{u}. \tag{2.12}$$

Changing the roles of the dependent and the independent variables, we get

$$\frac{dx}{du} = -\frac{u}{p(x)u + q(x)} = G(u, x). \tag{2.13}$$

The right hand side of (2.13) is a regular function of u and x near $(0, x_0)$, since $q(x_0) \neq 0$. We need an upper bound on $G(u, x)$ in an explicitly given

neighborhood of $(0, x_0)$. Let us remind that in order to simplify the statement of this and further bounds, we have introduced the following notations: $|q(x_0)| = \eta > 0$, $R = 2(|x_0| + 1)$, $r = min\left(\frac{1}{4}R, \frac{\eta}{2m(K+1)R^{m-1}}\right)$, where K as above is the maximum of the norms $\|p\|, \|q\|$. Finally, $M = \frac{4m(K+1)R^m}{\eta}$, $\delta = \frac{1}{M}$ and $c = \frac{4}{M\eta} = \frac{1}{m(K+1)R^m}$.

Proposition 2.1 *For $|u| \leq \delta$ and $|x - x_0| \leq r$ we have $|G(u,x)| \leq c$.*

Proof: Inside the disk D_R centered at the origin the derivative $q'(x)$ of the polynomial $q(x)$ satisfies $|q'(x)| \leq mKR^{m-1}$. By the choice of r we have $D_r(x_0) \subset D_R$. Hence for $x \in D_r(x_0)$,

$$|q(x)| \geq |q(x_0)| - r \cdot mKR^{m-1} \geq \frac{1}{2}\eta,$$

by the choice of r. On the other hand, inside D_R the inequality $|p(x)| \leq KR^m$ is satisfied, and for $|u| \leq \delta = \frac{1}{M} = \frac{\eta}{4m(K+1)R^m}$, we have $|p(x)u| \leq \frac{1}{4}\eta$. Hence the absolute value of the denominator $p(x)u + q(x)$ of $G(u,x)$ is at least $\frac{1}{4}\eta$ for $x \in D_r(x_0)$ and $|u| \leq \delta$. Under the same assumptions on x and u we finally obtain

$$|G(u,x)| \leq \frac{|u|}{|p(x)u + q(x)|} \leq \frac{1/M}{(1/4)\eta} = \frac{1}{m(K+1)R^m} = c.$$

This completes the proof of the proposition.

Returning to the proof of Theorem 2.1 we see that for $|u| \leq \delta$ and $x \in D_r(x_0)$ the following differential inequality is satisfied:

$$\frac{d|x(u)|}{dv} \leq c \tag{2.14}$$

in any normalized direction v. Hence for any u with $|u| \leq \delta$ (and assuming that x remains in $D_r(x_0)$) we obtain

$$|x(u) - x_0| \leq c\delta = \frac{1}{m(K+1)R^m} \cdot \frac{\eta}{4m(K+1)R^m} =$$

$$= \frac{\eta}{4(m(K+1)R^m)^2} < \frac{\eta}{4m(K+1)R^{m-1}} = \frac{1}{2}r.$$

Hence for $|u| \leq \delta$ the solution $x(x_0, u)$ of (2.13), satisfying $x(0) = x_0$, exists and it indeed remains in the disk $D_r(x_0)$. This justifies *a posteriori* the use of the differential inequality (2.14).

Therefore $x(x_0, u)$ is a regular analytic function of two complex variables u and x_0 defined in the domain $\Omega = \{x_0 \in \mathbb{C} \setminus \{x_1, \cdots, x_m\}, \quad |u| \leq \delta(x_0)\}$. Consequently, $x(x_0, u)$ can be represented by a converging power series

$$x(x_0, u) = A(x_0) + u \sum_{k=0}^{\infty} A_k(x_0) u^k, \tag{2.15}$$

with A, A_k analytic functions of x_0, univalued and regular in $\mathbb{C} \setminus \{x_1, \cdots, x_m\}$.

Let us show that in fact

$$x(x_0, u) = x_0 + u^2 \sum_{k=0}^{\infty} a_k(x_0) u^k, \quad a_0(x_0) \neq 0 \tag{2.16}$$

Indeed, the initial condition $x(x_0, u) = x_0$ implies $A(x_0) \equiv x_0$. Next, the equation (2.13) shows that the derivative $\frac{dx}{du}$ vanishes for $u = 0$ and hence there are no linear in u terms in (2.15). A direct computation shows that $a_0(x_0) = -\frac{1}{2q(x_0)}$ and hence $|a_0(x_0)| = \frac{1}{2\eta(x_0)}$. On the other hand, as it was shown above, for $|u| \leq \delta(x_0)$ the solution $x(x_0, u)$ of (2.13), satisfying $x(0) = x_0$, exists and it remains in the disk $D_r(x_0)$. Hence by the Cauchy formula for the Taylor coefficients we get $|a_k(x_0)| \leq r(x_0)\delta(x_0)^{-(k+2)}$.

Now the standard manipulations with the power series show that the solution $u(x) = u^{[x_0]}(x)$ of the equation (2.12), satisfying $u^{[x_0]}(x_0) = 0$, as a function of x has a ramification of order 2 at x_0, and it can be represented by a Puiseux series

$$u(x) = u^{[x_0]}(x) = \sum_{k=1}^{\infty} b_k(x_0)(x - x_0)^{\frac{k}{2}}, \; b_1(x_0) \neq 0, \tag{2.17}$$

converging in the disk $|x - x_0| \leq \xi(x_0)$, where $\xi(x_0)$ can be given explicitly through the parameters defined above. This completes the proof of Theorem 2.1. Note that the coefficients $b_k(x_0)$ are univalued and regular functions of x_0 in $\mathbb{C} \setminus \{x_1, \cdots, x_m\}$.

Corollary 2.2 *In a neighborhood of its singular point x_0, the solution $y(x) = y^{[x_0]}(x)$ of (1.1) is given by*

$$y(x) = y^{[x_0]}(x) = \frac{c(x_0)}{(x - x_0)^{1/2}} \cdot (1 + \sum_{k=1}^{\infty} c_k(x_0)(x - x_0)^{\frac{k}{2}}), \tag{2.18}$$

converging for $|x - x_0| \leq \xi_1(x_0)$, with $c(x_0) = \frac{1}{b(x_0)}$.

Proof:

$$y(x) = \frac{1}{u(x)} =$$

$$= \frac{1}{b(x_0)(x - x_0)^{1/2}} \cdot \frac{1}{1 + \sum_{k=1}^{\infty} \frac{b_{k+1}(x_0)}{b_1(x_0)}(x - x_0)^{k/2}} =$$

$$= \frac{1}{b(x_0)(x - x_0)^{1/2}} \left(1 + \sum_{k=1}^{\infty} c_k(x_0)(x - x_0)^{k/2}\right).$$

2.3 Singularities of $y(y_a, x)$ as functions of y_a

In order to relate the singularities of the Poincaré mapping with those of the solutions of (1.1), it is important to see how the initial value of the solution y at a certain regular point influences the position of the singularities of y. The description of the singularities of y given above, allows one to get a rather accurate information in this respect.

Let us fix a certain point $c \in \mathbb{C},\ c \neq x_1, \cdots, x_m$ (i.e. $q(c) \neq 0$).

Lemma 2.3 *For any y_c, sufficiently large in absolute value, the solution $y(y_c, x)$ of (1.1), satisfying $y(y_c, c) = y_c$, has a singularity $x_0 = x_0(y_c)$ in*

a neighborhood of c. The position of this singularity, $x_0(y_c)$, is a regular function of y_c for $|y_c|$ sufficiently large, and $\frac{dx_0}{dy_c} \neq 0$.

Proof: Rewrite the expression (2.16) representing the solution $x(x_0, u)$ of the equation (2.13), satisfying $x(x_0, 0) = x_0$, as $x(x_0, u) = x_0 + u^2 g(x_0, u)$. Here $g(x_0, u)$ is a regular function in in the domain $\Omega = \{x_0 \in \mathbb{C}\backslash\{x_1, \cdots, x_m\},\ |u| \leq \delta(x_0)\}$ defined above, and $g(x_0, 0) \neq 0$. Now let us require that $x(x_0, u) = c$, with c fixed. We get an equation

$$c = x_0 + u^2 g(x_0, u) = G(x_0, u) \tag{2.19}$$

between the position x_0 of the zero of u (i.e. of the singularity of $y = \frac{1}{u}$) and the value of u at c, $u = u(c) = \frac{1}{y(c)}$. Now for $u = 0$ and $x_0 = c$, $\frac{\partial G}{\partial x_0} = 1$. Hence by the implicit function theorem, x_0 is a regular function of $u = u(c)$, with $x_0(0) = c$. Moreover, $\frac{dx_0}{du} = -\frac{\frac{\partial G}{\partial u}}{\frac{\partial G}{\partial x_0}} \neq 0$ for small $u \neq 0$, since $\frac{\partial G}{\partial u} \neq 0$ for such u. This completes the proof.

Remark. Explicit constants can be given in the statement of Lemma 2.3, in the same terms as above.

Corollary 2.3 *Let the solution $y(y_a, x)$ of (1.1), satisfying $y(y_a, a) = y_a$ and continued to $x_0 \neq x_1, \cdots, x_m$ along a path s in $\mathbb{C} \setminus \{x_1, \cdots, x_m\}$, have a singularity at x_0. Then x_0 is a regular function of the initial value y_a, and $\frac{dx_0}{dy_a} \neq 0$.*

Proof: $y(y_a, x)$ tends to ∞ as x tends to x_0. Fix a point c on the path s, sufficiently close to x_0, such that for $y_c = y(y_a, c)$ and c the conditions of lemma 2.3 are satisfied. By this lemma, the singular point x_0 of the solution is a regular function of the initial value y_c, and $\frac{dx_0}{dy_c} \neq 0$. We have $\frac{dy_c}{dy_a} \neq 0$, the mapping $y_a \to y_c$ being the Poincaré mapping of (1.1) along the path s. We obtain that x_0 is a regular function of y_a, with $\frac{dx_0}{dy_a} = \frac{dx_0}{dy_c} \cdot \frac{dy_c}{dy_a} \neq 0$.

Using the equation (2.19) we can extend Lemma 2.3 above and describe the dependence of the position x_0 of the singular point of the solution on the

value of this solution at the "original singular point" itself. Let us fix, as above, a certain point $c \in \mathbb{C}$, $c \neq x_1, \cdots, x_m$ (i.e. $q(c) \neq 0$).

Proposition 2.2 *For y_c near ∞, the position $x_0(y_c)$ of the singularity of the solution $y(y_c, x)$ of (1.1), satisfying $y(y_c, c) = y_c$, can be represented by a convergent Taylor series in $u = u(c) = \frac{1}{y(c)}$*

$$x_0 - c = u^2 \sum_{k=0}^{\infty} \alpha_k u^k, \quad \alpha_0 \neq 0. \tag{2.20}$$

Conversely, the value $y(y_c, c) = y_c$ at c of the solution y of (1.1) having singularity at x_0, can be represented by a convergent fractional Puiseux series

$$u = u(c) = \sum_{k=1}^{\infty} \beta_k (x_0 - c)^{\frac{k}{2}}, \quad \beta_1 \neq 0. \tag{2.21}$$

Proof: We rewrite the equation (2.19) in the form

$$x_0 - c = -u^2 g(x_0, u) = -G(x_0, u) \tag{2.22}$$

between the position x_0 of the zero of u (i.e. of the singularity of $y = \frac{1}{u}$) and the value of u at c, $u = u(c) = \frac{1}{y(c)}$. Now, as above, for $u = 0$ and $x_0 = c$, $\frac{\partial G}{\partial x_0} = 1$. Hence by the implicit function theorem, $x_0 - c$ is a regular function of $u = u(c)$, with $x_0(0) - c = 0$. Therefore, $x_0 - c$ can be represented by the convergent Taylor series

$$x_0 - c = u^2 \sum_{k=0}^{\infty} \alpha_k u^k. \tag{2.23}$$

Solving the equation (2.23) with respect to u we get

$$u = u(c) = \sum_{k=1}^{\infty} \beta_k (x_0 - c)^{\frac{k}{2}}. \tag{2.24}$$

This completes the proof of the proposition. Below we shall use it to describe the ramification of the Poincaré mapping around its singularities.

3 Analytic continuation of the Poincaré mapping

In this section we give an accurate definition of the Poincaré mapping ϕ of the Abel Equation

$$y\prime = p(x)y^2 + q(x)y^3 \tag{1.1}$$

and discuss some problems related to the investigation of ϕ. Then we give a "semi-constructive" description of the analytic continuation of ϕ along a given path.

Let $a, b \in \mathbb{C},\ b \neq x_1, \cdots, x_m$, where $x_1, \cdots, x_m$ are, as above, all the zeroes of q. Notice that if a solution y of (1.1) happens to have a singularity at one of the $x_i,\ i = 1, \ldots, m$, the analytic structure of this solution near x_i may be much more complicated than that described in Section 2 above (see examples in Section 5 below). This is because the equation (2.13) has a singularity at $(0, x_i)$; both the numerator and the denominator of $G(u, x)$ vanish at this point.

Let s be a path in $\mathbb{C}$, joining a and b. We *do not assume* that s avoids the points $x_1, \cdots, x_m$, unless specifically stated. Let the initial value $y_a^0 \in \mathbb{C}$ be given. Assume that the solution $y(y_a^0, x)$ of the equation (1.1) satisfying $y(y_a^0, a) = y_a^0$ can be analytically continued along s from a neighborhood of a. In particular, this continuation does not have singularities on s.

Definition 3.1 *The (germ at y_a^0 of the) Poincaré mapping $\phi = \phi_{a,b,s,y_a^0}$ of the equation (1.1) along the path s is defined as follows: it associates to each y_a near y_a^0 the value y_b at the point b of the solution $y(y_a, x)$ of (1.1), satisfying $y(y_a, a) = y_a$, and continued to b along s. Thus $\phi(y_a) = y(y_a, b) = y_b$.*

Since $y \equiv 0$ is a solution of (1.1), the germ of the Poincaré mapping at zero satisfies $\phi(0) = 0$ along any path s and for any endpoints a, b. Moreover,

by Corollary 2.1, for any $R > 0$ the solutions $y(y_a, x)$ are regular in the disk D_R, assuming that $|y_a|$ is sufficiently small. Hence, for any a, b, s, the germ at the origin of $\phi_{a,b,s}$ is defined and it does not depend on the path s.

However, for larger values of y_a, the analytic continuation of $y(y_a, x)$ along different paths s may lead to different values of y_b.

Now assume that a path σ from $w_0 = \sigma(0)$ to $w_1 = \sigma(1)$ in the plane of the initial values y_a is given, parametrized by [0,1]. Assuming that none of the solutions $y(w_t, x),\ w_t = \sigma(t),\ t \in [0, 1]$, has a singularity on the path s, the definition above works well and defines the values (in fact, the germs) of $\phi(w_t),\ t \in [0, 1]$, and, in particular, $\phi(w_1)$,

The problem appears if the singularities of the solutions $y(w_t, x)$, continued along s, approach and cross the path s. The idea of the following construction is that if we can deform the path s (following the movement of w_t along the curve σ) in such a way that it escapes the singularities of $y(w_t, x)$, we can still use this deformed path for the analytic continuation of the solutions $y(w, x)$, and hence for the analytic continuation of ϕ.

Let σ as above be given. Assume that there exists a family $s_t,\ t \in [0, 1]$, of the paths from a to b, with the following properties:

1. s_t is a continuous in t deformation of the original path s, $s_0 = s$.

2. For each $t \in [0, 1]$, the solution $y(w_t, x)$, continued along s_t, is regular at each point of s_t.

Theorem 3.1 *The germ of the Poincaré mapping along s at the point w_0, ϕ_{a,b,s,w_0}, allows for the analytic continuation along σ from $w_0 = \sigma(0)$ to $w_1 = \sigma(1)$. The continued germ at w_1 is ϕ_{a,b,s_1,w_1} provided by the analytic continuation of the solutions starting near w_1 along s_1.*

Proof: We shall show that for each $t \in [0, 1]$ the value (the germ) of $\phi(w_t)$, obtained by an analytic continuation of ϕ along σ, is given by

$$\phi(w_t) = y(w_t, b), \tag{3.1}$$

with the solution $y(w_t, x)$, satisfying $y(w_t, a) = w_t$, being analytically continued from a to b along the path s_t. We can subdivide the process of the analytic continuation of ϕ into a finite number of small successive steps. In each step we first move w_t along σ, *not deforming* s_t (providing that the singularities of $y(w_t, x)$ do not hit s_t). Then we deform s_t, not changing w_t. Clearly, the first part of each step gives an analytic continuation of ϕ along σ, while the second step does not change ϕ at all. Therefore the total procedure provides the required analytic continuation of ϕ. This completes the proof of Theorem 3.1.

Theorem 3.1 reduces the problem of the analytic continuation of the Poincaré mapping ϕ to a construction of the family of paths s_t, with the properties stated above.

One particular case is very easy: if the singularities of $y(w_t, x)$ do not approach the path s, it does not need to be deformed, and we can take $s_t \equiv s$.

4 Singularities of the Poincaré mapping

Let s be a path in $\mathbb{C}$ joining two points a and b, and let the initial value $y_a^0 \in \mathbb{C}$ be given. Assume that the solution $y(y_a^0, x)$ of the equation (1.1) satisfying $y(y_a^0, a) = y_a^0$ can be analytically continued along s from a neighborhood of a to each point of s except, possibly, the endpoint b. In particular, this continuation does not have singularities in the interior points of s. If b is also a regular point of this solution, then the germ at y_a^0 of the Poincaré mapping $\phi = \phi_{a,b,s,y_a^0}$ along the path s is defined and regular.

Consider now the case when the analytic continuation along s of the solution $y(y_a^0, x)$ has a singularity at b. From now on, we assume that b is different from the fixed singularities $x_1, \ldots, x_m$.

Proposition 4.1 *Under the above assumptions there is a germ of a real*

curve $\gamma \subset \mathbb{C}$ at y_a^0 such that for $y_a \notin \gamma$ the analytic continuation along s of the solution $y(y_a, x)$ is regular at each point of s including the endpoint b

Proof: We are in a situation of Corollary 2.3 above. By this corollary, the position $x_0(y_a)$ of the singularity of the solution $y(y_a, x)$ near b is a regular function of y_a near y_a^0. So the curve γ is formed by exactly those y_a for which this singularity $x_0(y_a)$ belongs to s. By the description of the singularities of the solutions of (1.1) given in Section 2, $x_0(y_a)$ is the only singular point of the local branch of the solution $y(y_a, x)$ near b. On the other side, by the assumptions, there are no singularities of the solution $y(y_a, x)$ on s not in a neighborhood of b. Therefore, for $y_a \notin \gamma$ the analytic continuation along s of the solution $y(y_a, x)$ is regular at each point of s, including the endpoint b. This completes the proof of the proposition.

Now we are ready to describe the generic singular points of the Poincaré mapping.

Theorem 4.1 *Let s be a path in $\mathbb{C}$ joining two points a and $b \neq x_1, \ldots, x_m$, and let the initial value $y_a^0 \in \mathbb{C}$ be given. Assume that the solution $y(y_a^0, x)$ of the equation (1.1) satisfying $y(y_a^0, a) = y_a^0$ can be analytically continued along s from a neighborhood of a to each point of s except the endpoint b, where this solution has a singularity. Then for each y_a in a neighborhood of y_a^0, such that $y_a \notin \gamma$, where the curve γ has been defined in Proposition 4.1, the germ at y_a of the Poincaré mapping $\phi = \phi_{a,b,s,y_a}$ along the path s is defined and regular. In a punctured neighborhood U_0 of y_a^0 these germs can be analytically continued across γ, to form a double-valued regular in U_0 function ϕ, which allows for a representation by a Puiseux series*

$$y(b) = \frac{1}{\sqrt{(y_a - y_a^0)}} \sum_{k=0}^{\infty} \nu_k (y_a - y_a^0)^{\frac{k}{2}}, \quad \nu_0 \neq 0, \tag{4.1}$$

convergent in U_0.

Proof: The fact that for each y_a in a neighborhood of y_a^0, such that $y_a \notin \gamma$, the germ at y_a of ϕ is defined and regular, follows directly from Proposition 4.1. A possibility of the analytic continuation of ϕ across γ follows from the results of Section 3. The local form of ϕ near y_a^0 can be obtained in two ways.

The first one uses Corollary 2.1 and Proposition 2.2. By Corollary 2.1, the position $x_0(y_a)$ of the singularity (near b) of the solution $y(y_a, x)$, analytically continued along s, is a regular function of y_a. By Proposition 2.2, $u_b = \frac{1}{y_b}$, where $y_b = y(y_a, b)$, as a function of x_0, is given by

$$u = u_b = \sum_{k=1}^{\infty} \beta_k (x_0 - b)^{\frac{k}{2}}, \quad \beta_1 \neq 0. \tag{4.2}$$

Substituting into this expression a regular function $x_0(y_a)$, $x_0(y_a^0) = b$, we get

$$u = u_b = \sum_{k=1}^{\infty} c_k (y_0 - y_a^0)^{\frac{k}{2}}, \quad c_1 \neq 0. \tag{4.3}$$

Finally, expressing $y_b = \frac{1}{u_b}$ through u_b via (4.3), we get the required formula (4.1).

5 Analysis of a local model

Here, we want to investigate the local behaviour of the solutions near a generic fixed singularity. That is to say, we assume that the polynomial $q(x)$ has a simple zero and we indeed replace the equation by the following:

$$\frac{dy}{dx} = cxy^3 + y^2. \tag{5.1}$$

The change of unknown function

$$y = \frac{v}{x},$$

yields

$$-\frac{v}{x^2} + \frac{v'}{x} = \frac{cv^3}{x^2} + \frac{v^2}{x^2},$$

and thus:

$$v' = \frac{1}{x}[cv^3 + v^2 + v], \tag{5.2}$$

which obviously separates.

Write:

$$\frac{1}{cv^3 + v^2 + v} = \frac{\alpha}{v} + \frac{\beta}{v - v_1} + \frac{\gamma}{v - v_2},$$

with:

$$v_1 = \frac{-1 + \sqrt{1 - 4c}}{2c}, \quad v_2 = \frac{-1 - \sqrt{1 - 4c}}{2c},$$

and

$$\alpha = 1, \quad \beta = \frac{1}{cv_1(v_1 - v_2)}, \quad \gamma = \frac{1}{cv_2(v_2 - v_1)}.$$

Integrating equation (5.2) we get for each its solution $v(x)$

$$v(v - v_1)^\beta (v - v_2)^\gamma = K \cdot x,$$

for a certain constant K. Equivalently

$$y(xy - v_1)^\beta (xy - v_2)^\gamma = K,$$

or

$$y(1 - \frac{xy}{v_1})^\beta (1 - \frac{xy}{v_2})^\gamma = \frac{K}{v_1^\beta v_2^\gamma} = K'. \tag{5.3}$$

Notice that the only "fixed singularity" of the equation (5.1) is $x = 0$. To start with, let us take this point $x = 0$ as the initial point a. Now, the constant K' in (5.3) is evaluated by setting $x = 0$ and $y = y_0$ and this yields $K' = y_0$. Therefore, the solution $y(y_0, x)$ of (5.1) satisfying $y(y_0, 0) = y_0$ is given by

$$y(x)(1 - \frac{xy(x)}{v_1})^\beta (1 - \frac{xy(x)}{v_2})^\gamma = y_0. \tag{5.4}$$

Substituting into (5.4) the point $x = b$, we get the relation between $y_b = y(b)$ and y_0 in the form:

$$y_0 = y(1 - \frac{by_b}{v_1})^\beta (1 - \frac{by_b}{v_2})^\gamma. \tag{5.5}$$

Now, we are interested in the "limit cycles" of the equation (5.1), i.e. in its solutions $y(x)$ satisfying $y(0) = y(b)$. This relation together with (5.5) gives the equation for the limit cycles, which are (besides the solution $y \equiv 0$ of (5.1)) in one-to-one correspondence with the solutions of:

$$(1 - \frac{by_0}{v_1})^{\beta}(1 - \frac{by_0}{v_2})^{\gamma} = 1. \tag{5.6}$$

At this point we have to clarify the geometric interpretation of the "limit cycles", as defined above. The problem is that the solutions of (5.1) are multivalued functions. The accurate interpretation of the equation (5.6) is that *the algebraic curve $Y = Y_{y_0}$, defined by (5.4), passes through the points $(0, y_0)$ and (b, y_0).* Certainly, this curve Y, parametrized as $y = y(x)$, satisfies differential equation (5.1). But a priori we do not even know whether Y is connected. So (5.6) by itself does not exclude a possibility that the points $(0, y_0)$ and (b, y_0) belong to different leaves of the solutions of the differential equation (5.1).

Below we show that in fact for $y_0 \neq 0$ the curve $Y = Y_{y_0}$ is connected. This allows us to give the following interpretation to the equation (5.6): *for each y_0 satisfying (5.6) there exists a path s from 0 to b such that the solution $y(y_0, x)$ can be analytically continued along s, and this continuation satisfies $y(y_0, b) = y_0$.*

We now choose some specific values for the free parameter c in order to bring some light on possible solutions of (5.6). Assume that

$$1 - 4c = \delta^2,$$

where $\delta = -\frac{1}{2n+1}$ with n an integer. In that case we obtain:

$$\beta = n, \ \gamma = -n - 1,$$

(see Section 6 below for more detailed computations). Limit cycles of equation (5.1), in the interpretation given above, are in correspondence with the

solutions of:

$$(1-\frac{by_0}{v_1})^n = (1-\frac{by_0}{v_2})^{n+1}. \tag{5.7}$$

One can easily show that for large integer values of n equation (5.7) has n distinct complex solutions $y_0^j,\ j=1,\dots,n.$ (see Section 7 below). Consider the local solutions $y^j(x)=y(y_0^j,x)$ at the origin satisfying $y^j(0)=y_0^j$. Combining equation (5.7) with Theorem 6.1 below which describes the monodromy of the solutions of (5.1) we get the following result:

Theorem 5.1 *For $c=\frac{1}{4}(1-\frac{1}{(2n+1)^2})$ and $b\neq 0$ equation (5.1) has n different "limit cycles", i.e. local solutions $y^j(x)$ at the origin, $j=1,\dots,n$, and paths s^j from 0 to b, such that each $y^j(x)$ being analytically continued along s^j satisfies $y(0)=y(b)$.*

The proof of this theorem is given at the end of Section 6 below. From the description given in Section 6 it follows that the paths s^j have the following form: s^j goes from zero to the (only) singularity of $y^j(x)$, turns once around this singularity, returns to zero, makes $m\le n$ turns around zero, and finally comes to b.

Therefore, in this example we see that the equation (5.1) may have as many complex limit cycles as we wish, when n is increased, although the degree of the coefficients of this equation remains bounded. This phenomenon reminds (in much simpler setting) the counterexample due to Yu. Iliashenko of Petrowski-Landis claim ([15]). It should also be compared with the examples of differentiable Abel equations discussed by A. Lins Neto ([18]). Note that Khovanski fewnomials theory, or, rather, "additive complexity" arguments (see [**?**, 26]) imply that the number of real roots of equation (5.7) remains bounded independently of n. So that this example does not provide a counterexample to real Hilbert-Pugh problem.

Remark. One can investigate the situation for another choice of the parameter c. If we put $\frac{n}{n+1}=k$ and let k to be a large integer, this corresponds to

$n \approx -1$ or $c \approx 0$. The equation (5.4) takes the form

$$(1-\frac{xy}{v_1})^k = y_0^{\mu}(1-\frac{xy}{v_2}). \tag{5.8}$$

while the equation (5.7) takes the form

$$(1-\frac{by_0}{v_1})^k = (1-\frac{by_0}{v_2}). \tag{5.9}$$

The investigation of this case may be important since as c tends to zero, equation (5.1) tends to the integrable equation $y' = y^2$. We now consider the case $c = \frac{1}{4}$ and the equation

$$\frac{dy}{dx} = \frac{1}{4}xy^3 + y^2.$$

With $y = v/x$, this equation yields:

$$\frac{dv}{dx} = \frac{1}{4x}(v^3 + 4v^2 + 4),$$

which separates and gives the solution $y(x)$ corresponding to the initial data y_0 as the solution to:

$$\frac{y}{xy+2}\mathrm{e}^{\frac{2}{xy+2}} = \frac{\mathrm{e}}{2}y_0.$$

Periodic orbits correspond to solutions of $y(1) = y(0) = y$ to

$$\frac{2}{y+2}\mathrm{e}^{\frac{2}{y+2}} = \mathrm{e}.$$

If we change $y = \frac{2\xi}{1-\xi}$, this yields

$$1 - \xi = \mathrm{e}^{\xi}.$$

We write $\xi = x + \mathrm{i}y$, and derive the two equations

$$1 - x = \mathrm{e}^x \mathrm{cos} y,$$

$$-y = \mathrm{e}^x \mathrm{sin} y.$$

Note that if (x, y) is a solution, then $(x, -y)$ is also a solution. Then we can assume $y > 0$. Second equation implies $\sin(y) < 0$ and we restrict ourselves now to $y \in](2n+1)\pi, (2n+2)\pi[$. Now we plug $x = -\log(-\frac{\sin y}{y})$ into the first equation. This displays:

$$F(y) = 1 + \log(-\frac{\sin y}{y}) + \frac{y}{\tan y} = 0.$$

Then we note that as $y \to (2n+1)\pi$, $F(y) \to +\infty$ and that as $y \to (2n+2)\pi$, $F(y) \to -\infty$. There is thus at least one solution (and in fact a single one) in the interval. The Abel equation has thus infinitely many limit cycles.

The basic example (5.1) can be used to generate a family of similar examples by composition. Composition appears quite naturally in the subject (see both the classics (Abel,Liouville,...) and more recent contributions ([4]-[8]). Consider the Abel equations of the form

$$y' = cP(x)p(x)y^3 + p(x)y^3, \tag{5.10}$$

where $p(x)$ is an arbitrary polynomial, and $P(x)$ is the anti-derivative of $p(x)$ which vanishes at $x = 0$. The change of variables $w = P(x)$ brings (5.9) to the form

$$\frac{dy}{dw} = cwy^3 + y^2. \tag{5.11}$$

Applying the above given analysis of this last equation, we see that the solution $y(x)$ to the equation (5.9) satisfying $y(0) = y_0$, solves the implicit algebraic equation:

$$y(1 - \frac{P(x)y}{v_1})^{\beta}(1 - \frac{P(x)y}{v_2})^{\gamma} = y_0. \tag{5.12}$$

Hence also the limit cycles of (5.9) can be investigated in a similar way. Notice, however, that a special composition structure of the solutions of (5.9), namely, that each its solution y can be represented as $y(x) = \tilde{y}(P(x))$, for $\tilde{y}(w)$ solving (5.10), implies the following: for each a, b with $P(a) = P(b)$ we have $y(a) \equiv y(b)$.

6 Ramification of solutions of $\frac{dy}{dx} = cxy^3 + y^2$

To study in detail the ramification of the solutions of the Abel equation (5.1), $\frac{dy}{dx} = cxy^3 + y^2$, we choose the parameter c in this equation in the same way as above. We would like c to tend to $\frac{1}{4}$, which is a “discriminant point” for the denominator $cv^3 + v^2 + v$ appearing after separation of variables in (5.1). On the other hand, we want the first integral to remain algebraic. So let us write

$$1 - 4c = \delta^2, \quad c = \frac{1}{4} - \frac{\delta^2}{4}$$

where δ is small. In this case we obtain:

$$v_1 = -\frac{2}{1+\delta} \approx -2 + 2\delta, \quad v_2 = -\frac{2}{1-\delta} \approx -2 - 2\delta.$$

For β and γ we get, respectively,

$$\beta = -\frac{1+\delta}{2\delta}, \quad \gamma = \frac{1-\delta}{2\delta}.$$

Notice, that $\beta + \gamma = -1$. Thus, taking $\delta = -\frac{1}{2n+1}$ for n an integer, we obtain $\beta = n, \quad \gamma = -n-1$. Let us fix this choice of c.

The first integral (5.4) takes now the form

$$y_0 = \frac{y(1 - \frac{xy}{v_1})^n}{(1 - \frac{xy}{v_2})^{n+1}} = H(x, y). \tag{6.1}$$

In the rest of this section we investigate in some detail the solutions of the algebraic equation (6.1). First of all, we notice that for $y_0 = 0$ we indeed get *two separate leaves*: the straight line $Y_0^0 = \{y \equiv 0\}$ and the hyperbola $Y_0^1 = \{y = \frac{v_1}{x}\}$. As the zero curves of $H(x, y)$ the first has the multiplicity 1, while the second has the multiplicity n.

The second remark is that the pole locus of $H(x, y)$ is the hyperbola $Y_\infty = \{y = \frac{v_2}{x}\}$ which has the multiplicity $n + 1$. As we can expect, both the hyperbolas above are solutions of the differential equation (5.1). This can be checked by a direct substitution.

6.1 Critical points of H(x,y)

Let us find the critical points of the function $H(x,y)$.

Lemma 6.1 *All the critical points of the function $H(x,y)$ (in fact, all the points with $H_y(x,y)=0$) are situated on the hyperbola $Y_0^1=\{y=\frac{v_1}{x}\}$.*

Proof: After differentiating H with respect to x and y, cancelling the common degrees of $(1-\frac{xy}{v_2})$, equating the numerator to zero, and some computations, using, in particular, the identities

$$\frac{n}{v_2}-\frac{n+1}{v_1}=0,\quad \frac{n}{v_1}-\frac{n+1}{v_2}=1,$$

we obtain the following system of equations:

$$(1-\frac{xy}{v_2})^n H_y=(1-\frac{xy}{v_1})^{n-1}=0,$$

$$(1-\frac{xy}{v_2})^n H_x=(1-\frac{xy}{v_1})^{n-1}(1-\frac{xy}{v_1v_2})=0.$$

The common zeroes of this system lie exactly on the parabola $Y_0^1=\{y=\frac{v_1}{x}\}$. Notice that the partial derivative H_x vanishes, in addition, on the hyperbola $y=\frac{v_1v_2}{x}$. Hence, the points of this hyperbola are zeroes of the derivative y' of the solutions of (5.1) passing through these points. Of course, this can be checked by the direct substitution.

Remark 1. The fact that H is the first integral of the equation (5.1), and hence its level curves must be locally graphs of a regular function at each finite point, does not exclude by itself possible critical points of H - compare the points of the hyperbola Y_0^1.

Remark 2. Instead of the rational equation (6.1) we can consider the equivalent polynomial equation

$$y(1-\frac{xy}{v_1})^n-y_0(1-\frac{xy}{v_2})^{n+1}=0. \tag{6.2}$$

The advantage of (6.1) is that the initial value y_0 appears there just as the right hand side.

Now, differentiating (6.2) we get the following system of equations:

$$(1-\frac{xy}{v_1})^n + \frac{nxy}{v_1}(1-\frac{xy}{v_1})^{n-1} - \frac{(n+1)xy_0}{v_2}(1-\frac{xy}{v_2})^n = 0,$$
$$\frac{ny^2}{v_1}(1-\frac{xy}{v_1})^{n-1} - \frac{(n+1)yy_0}{v_2}(1-\frac{xy}{v_2})^n = 0.$$

Multiplying the first equation by y and the second by x and taking the difference, we get

$$y(1-\frac{xy}{v_1})^n = 0.$$

So either $y=0$ or the point (x,y) belongs to Y_0^1. If $y=0$ the second equation above is satisfied, while the first equation gives $x = \frac{v_2}{(n+1)y_0}$. So the equation (6.2) has an additional critical point, not on the hyperbola Y_0^1. Notice, however, that for any $y_0 \neq 0$ this point does not belong to the solution curve of (6.2), while for $y_0 = 0$ it is at infinity.

6.2 Singularities of solutions of $\frac{dy}{dx} = cxy^3 + y^2$

The only "fixed" singularity of the equation (5.1) is the origin $x=0$. Let us start with the "movable" singularities $x_0 \neq 0$ of the solutions (compare with the general results of Section 2).

Proposition 6.1 *For $y_0 \neq 0$ the solution $y(y_0,x)$ has the only movable singularity at the point $x_0(y_0) = \frac{\kappa}{y_0}$, where $\kappa = -v_2(\frac{v_2}{v_1})^n = -\frac{2}{1-\delta}(\frac{1-\delta}{1+\delta})^n \approx -2e$ for δ small. Exactly one local branch of $y(y_0,x)$ takes an infinite value and has a ramification of order 2 at $x_0(y_0)$, while the other $n-1$ local branches are regular at this point and take there $n-1$ different finite values.*

Proof: Denoting, as above, $\frac{1}{y}$ by u we get from (6.1)

$$y_0 = \frac{(u-\frac{x}{v_1})^n}{(u-\frac{x}{v_2})^{n+1}}. \qquad (6.3)$$

Substituting here $u = 0$ (and assuming $x \neq 0$ and so cancellation is possible) we get $x_0(y_0) = \frac{\kappa}{y_0}$. To get the series expansions we rewrite (6.3) as follows:

$$x = \frac{\kappa}{y_0} \frac{(1 - \frac{uv_1}{x})^n}{(1 - \frac{uv_2}{x})^{n+1}} = \frac{\kappa}{y_0}(1 + \frac{u}{x}[(n+1)v_2 - nv_1] + A(\frac{u}{x})^2 + \ldots). \tag{6.4}$$

Since $(n + 1)v_2 - nv_1 = 0$, we can rewrite the last expression as

$$x = \frac{\kappa}{y_0}(1 + A(\frac{u}{x})^2 + \ldots), \quad x - \frac{\kappa}{y_0} = B(\frac{u}{x})^2(1 + \ldots). \tag{6.5}$$

This shows that $x - \frac{\kappa}{y_0}$ as a function of u has a second order zero at $u = 0$, and hence u as a function of x has at $x_0 = \frac{\kappa}{y_0}$ a second order branching. (We do not prove that the coefficient B above is different from zero, since this fact was shown in general form in Section 2 above). This completes the description of the branch passing through the point $(x_0(y_0), \infty)$.

Each other branch of the solution $y(y_0, x)$ (i.e. of the curve $H(x, y) = y_0$) over x_0 is regular, since by Lemma 6.1 all the singularities of H belong to the level curve $H = 0$. Since for any fixed x the total number of the solutions of $H(x, y) = y_0$ with respect to y, counted with multiplicities, is $n+1$, and since the multiplicity of the singular branch is 2, there are exactly $n - 1$ regular local branches of the curve $H(x, y) = y_0$ over x_0. This completes the proof of the proposition.

Remark. Exactly as in Sections 3 and 4 above, we can use the series (6.5) to analyze the local structure of singularities of the Poincaré mapping. Indeed, for a fixed $x = x_0$ we can rewrite (6.5) as

$$y_0 - \frac{\kappa}{x_0} = D(\frac{u}{x_0})^2(1 + \ldots), \tag{6.6}$$

and we get a second order zero of $y_0 - \frac{\kappa}{x_0}$ as a function of u and a second order ramification of u as a function of y_0.

The next step is to investigate the structure of the fixed singularity $x = 0$.

Proposition 6.2 *For $y_0 \neq 0$ the solution $y(y_0, x)$ has over $x = 0$ two local components: the regular one, passing through the point $(0, y_0)$, and the singular one, passing through the point $(0, \infty)$. The singular component is represented by the Puiseux series*

$$\frac{1}{y(x)} = u(x) = \frac{1}{v_1}x + \mu x^{1+\frac{1}{n}} + \ldots, \tag{6.7}$$

with

$$\mu = y_0^{\frac{1}{n}}(\frac{1}{2n+1})^{1+\frac{1}{n}} \approx \frac{1}{2n}y_0^{\frac{1}{n}}.$$

In particular, the local monodromy acts as a cyclic permutation of the infinite branches.

Proof: Let us rewrite the equation (6.3) in the form

$$y_0(u - \frac{x}{v_2})^{n+1} = (u - \frac{x}{v_1})^n. \tag{6.8}$$

We have to find the Puiseux expansion of the curve given by (6.8) at the point $(u, x) = (0, 0)$. To simplify the presentation, we use the following "einsatz":

$$u(x) = \frac{1}{v_1}x + \mu x^\nu + \ldots. \tag{6.9}$$

Substituting (6.9) to (6.8) we get

$$y_0[(\frac{1}{v_1} - \frac{1}{v_2})x + \mu x^\nu + \ldots]^{n+1} = \mu^n x^{\nu n} + \ldots.$$

Comparing the leading degrees and coefficients, we obtain

$$y_0(\frac{1}{v_1} - \frac{1}{v_2})^{n+1}x^{n+1} + \ldots = \mu^n x^{\nu n} + \ldots,$$

and hence

$$\mu = y_0^{\frac{1}{n}}(\frac{1}{v_1} - \frac{1}{v_2})^{\frac{n+1}{n}} = y_0^{\frac{1}{n}}(-\delta)^{1+\frac{1}{n}} \approx \frac{1}{2n}y_0^{\frac{1}{n}}, \quad \nu = 1 + \frac{1}{n}.$$

6.3 Global ramification of solutions

According to Proposition 6.1, there are only two singularities of the solution $y(y_0, x)$: the fixed singularity at $x = 0$ and the movable singularity at $x_0(y_0) = \frac{\kappa}{y_0}$. The original local branch at $x = 0$ of $y(y_0, x)$ is regular at the origin. Hence, it can be analytically extended as a regular univalued function into the disk $\mathcal{D} = D_{|\frac{\kappa}{y_0}|}$, centered at $x = 0$.

Lemma 6.2 *The regular branch of $y(y_0, x)$ on the disk $\mathcal{D}$ has a singularity at the boundary point $\frac{\kappa}{y_0}$.*

Proof: Take y_0 positive. By our choice of the parameter $c \approx \frac{1}{4}$ we have $c > 0$. Therefore, the right hand side of (5.1) is positive, and bounded from below by y^2, and hence its solution blows up in finite time on the semi-axis $x > 0$. By Proposition 6.1, this happens exactly at the point $x_0(y_0) = \frac{\kappa}{y_0}$. This proves Proposition 6.2 for y_0 positive. Now, as y_0 moves along the circle $|y_0| = Const$, the singularity $x_0(y_0) = \frac{\kappa}{y_0}$ of the regular univalued function $y(y_0, x)$ on the disk $\mathcal{D}$ moves along the boundary of this disk. Since $y(y_0, x)$ analytically depends on y_0, the point $x_0(y_0)$ remains its singularity. This completes the proof.

Let the value $y_0 \neq 0$ be fixed. Consider the loop ω following the straight segment from 0 to the singular point $x_0(y_0)$, then going around this point in a counter-clockwise direction along a small circle, and then returning to 0 along the same straight segment.

Lemma 6.3 *The regular branch at $x = 0$ of the solution $y(y_0, x)$ analytically continued along the loop ω, returns at $x = 0$ to one of the infinite branches of the solution.*

Proof: Since $y(y_0, x)$ has a second order ramification at x_0, after one turn around this point we get *another* branch of the solution. As we return to zero, we stay on this new branch, different from the initial (regular) one.

But by Proposition 6.2 all the branches, except the initial one, tend to ∞ at $x = 0$.

Now we have enough tools to prove one of the main properties of the solutions of (5.1), as given by the first integral $H(x, y) = y_0$:

Theorem 6.1 *For each $y_0 \neq 0$ the solution curve $Y_{y_0} = \{H(x, y) = y_0\}$ is irreducible. The analytic continuation of the local solution $y(y_0, x)$ at zero along the loop ω and then several turns around zero transform this local branch to each one of the n remaining branches of Y_{y_0}.*

Proof: By Lemma 6.3 continuation along ω transforms the local regular branch of $y(y_0, x)$ at zero into one of the infinite branches. By proposition 6.2, each turn around zero results in a cyclic permutation of the n infinite branches. Hence, in at most n turns each other infinite branch can be obtained.

Remark. Another proof of Theorem 6.1 can be obtained by computing the ramification of Y_{y_0} at $x = \infty$. Rewriting the equation (6.3) in the form

$$y_0\left(\frac{1}{y} - \frac{x}{v_2}\right)^{n+1} = \left(\frac{1}{y} - \frac{x}{v_1}\right)^n,$$

and then substituting $x = \frac{1}{w}$, we obtain

$$\kappa w y (y - v_1 w)^n = y_0 (y - v_2 w)^{n+1}. \tag{6.10}$$

The einsatz $y = v_2 w + \eta w^\rho + \ldots$ leads to

$$\kappa w (v_2 w + \eta w^\rho + \ldots)[(v_2 - v_1)w + \eta w^\rho + \ldots]^n = y_0 (\eta w^\rho + \ldots)^{n+1},$$

which produces, via comparing the leading terms,

$$\kappa v_2 (v_2 - v_1)^n w^{n+2} = y_0 \eta^{n+1} w^{\rho(n+1)}$$

and

$$\eta = \left(\frac{\kappa v_2}{y_0}\right)^{\frac{1}{n+1}} (v_2 - v_1)^{\frac{n}{n+1}}, \quad \rho = 1 + \frac{1}{n+1}.$$

We conclude that all the $n+1$ branches of Y_{y_0} tend to zero at $w=0$ or $x=\infty$, and that the local monodromy around infinity produces a cyclic permutation of these $n+1$ branches.

Proof of Theorem 5.1. Let us remind equation (5.7):

$$(1-\frac{by_0}{v_1})^n = (1-\frac{by_0}{v_2})^{n+1}.$$

One has to show that for large integer values of n equation (5.7) has n distinct complex solutions y_0^j, $j=1,\dots,n$. Consider the local solutions $y^j(x)=y(y_0^j,x)$ at the origin satisfying $y^j(0)=y_0^j$. The analytic continuation of $y^j(x)$ gives an algebraic curve $Y_{y_0^j}$ satisfying equation (5.4) with $y_0=y_0^j$. Now equation (5.7) says exactly that the points $(0,y_0^j)$ and (b,y_0^j) belong to $Y_{y_0^j}$. By Theorem 6.1 this curve is irreducible, and we can pass from the point $(0,y_0^j)$ to the point (b,y_0^j) via the analytic continuation of the local branch $y^j(x)=y(y_0^j,x)$ at the origin along the path s^j, as described in Theorem 5.1. This completes the proof.

References

[1] N. Abel, *Oeuvres complètes.* II S.Lie and L. Sylow, Edts., Christiana (1881).

[2] N. N. Bautin, *On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type*, Mat. Sb. **30** (1952), 181-196; Amer. Math. Soc. Transl. **5** (1962), 396-413.

[3] M. Blinov, M. Briskin, Y. Yomdin, *Local center conditions for a polynomial Abel equation*, to appear in Journal d'Anal.Math.

[4] M. Briskin, J.-P. Francoise and Y. Yomdin, "The Bautin ideal of the Abel equation", Nonlinearity, **11**, (1998), 431-443.

[5] M. Briskin, J.-P. Francoise and Y. Yomdin, “Center conditions, compositions of polynomials and moments on algebraic curves” Ergodic Theory Dyn. Syst. **19**, 5, (1999), 1201-1220.

[6] M. Briskin, J.-P. Francoise and Y. Yomdin, “Center condition II: Parametric and model center problems”, Isr. J. Math. **118**, (2000), 61-82.

[7] M. Briskin, J.-P. Francoise and Y. Yomdin, “Center condition III: Parametric and model center problems”, Isr. J.Math. **118**, (2000), 83-108.

[8] M. Briskin, J.-P. Francoise and Y. Yomdin, “Generalized moments, center-focus conditions and compositions of polynomials” in “*Operator theory, system theory and related topics*”, Oper. Theory Adv.Appl., bf 123, (2001), 165-185.

[9] M. Briskin, N. Roytvarf and Y. Yomdin, Center Conditions at Infinity, in preparation.

[10] J.-P. Francoise, Local bifurcations of limit cycles, Abel equations and Liénard systems, Notes by Hur Seok, in *Normal Forms, Bifurcations and Finiteness Problems in Differential Equations*, Edited by Y. Il-iashenko, C. Rousseau and G. Sabidussi, Nato Science Series, II. Mathematics, Physics and Chemistry-Vol. 137, Kluwer Academic Press, 187-209 (2004).

[11] J.-P. Francoise, Y. Yomdin, *Bernstein inequality and applications to analytic geometry and differential equations*, J. of Functional Anal., **146**, (1997), no.1, 185-205.

[12] J.-P. Francoise, Y. Yomdin, *Projection of analytic sets and Bernstein inequalities*, in ”Singularities Simposium - Lojasiewicz 70”, B. Jakubczyk, W. Pawlucki and J. Stasica (eds.), Banach Center Publications, Vol. 44, Warszawa 1998, 103-108.

[13] V. Golubev, Lectures on Analytic Theory of Differential Equations-Gostehizdat, Moscow, 1950.

[14] E. L. Ince, Ordinary Differential Equations, Dover, 1949.

[15] Yu. Ilyashenko, *An example of equations* $dw/dz = P_n(z,w)/Q_n(z,w)$ *having a countable number of limit cycles and arbitrarily high Petrowski-Landis genus* Mat. Sb. **122** (3) (1969), 388-404.

[16] F. Loray, Sur les theoremes I et II de Painlevé, arXiv:math.CA/0403034 v1, March 2004.

[17] G.M. Murphy, *"Ordinary Differential Equations and their solutions"*, Van Nostrand, Princeton, 1960.

[18] A.Lins Neto, "On the number of solutions of the equation x' = P(x,t) for which x(0) = x(1)", *Inventiones Math.* **59** (1980), 67-76.

[19] R. Liouville, "Sur une équation différentielle du premier ordre", Acta Mathematica, **27**, (1903), 55-78.

[20] C. Li, W. Li, J. Llibre, Z. Zhang, *New families of centers and limit cycles for polynomial differential systems with homogeneous nonlinearities* Ann. Differential Equations, **19**, n 3, 302-317 (2003).

[21] E.S. Cheb-Terrab, A.D. Roche "Abel Equations: Equivalence and Integrable Classes". Computer Physics Communications, **130**, (2000), 197-

[22] A.G. Khovansky, *Fewnomials*, AMS Publ., Providence, RI, (1991).

[23] P. Painlevé, Oeuvres completes. Editions du C.N.R.S. (R. Gerard, G. Reeb et A. Sec) Paris, 1972.

[24] F. Pakovich, N. Roytvarf and Y. Yomdin, *Cauchy type integrals of Algebraic functions and a tangential Center-Focus problem for Abel Equations*, to appear, Isr. J. of Math.

[25] I. Petrowski, E. Landis, *On the number of limit cycles of the equation* $dy/dx = P(x, y/Q(x, y)$ *where* $P(x, y)$ *and* $Q(x, y)$ *are polynomials of* 2^{nd} *degree*, Mat.Sb. **37** (1955), 209-250.

[26] J.-J. Risler, *Additive Complexity and zeros of real polynomials*, SIAM Journal of Comput. **14:1**, 178-183 (1985).

[27] Y. Yomdin, *Global finiteness properties of analytic families and algebra of their Taylor coefficients*, Proceedings of the Arnoldfest, Toronto, 1997, Fields Inst. Communications, Vol. 24, AMS, Providence, RI, (1999), 527-555.